

Orbital magnetic moment of the electron in the hydrogen atom in deformed space with minimal length

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Abstract

We investigated the orbital magnetic moment of electron in the hydrogen atom in deformed space with minimal length. It turned out that corrections to the magnetic moment caused by deformation depend on one parameter in the presence of two-parametric deformation. It is interesting to note that the correction to orbital magnetic moment is similar to the correction that follows from relativistic theory but it has an opposite sign. Using the upper bound for minimal length obtained in previous papers we estimated the upper bound for relative correction to orbital magnetic moment and obtained the value $\sim 10^{-12}$. This is four power less than the relative error for most recent experimental values of Bohr magneton.

1 Introduction

In recent years a lot of attention has been devoted to quantum mechanics with a deformed commutation relations. This interest was impelled by several independent lines of investigation such as string theory and quantum gravity which suggested the existence of a finite lower bound to the possible resolution of length (minimal length) [1, 2, 3]. Kempf *et al.* showed that minimal length could be obtained as a minimal uncertainty in position from the deformed commutation relations [4, 5, 6, 7, 8]. In [9] it was shown that generalized commutation relations leading to the existence of minimal length could be obtained from modified dispersion relations. We also note that for the first time the deformed algebra leading to a quantized space-time was introduced by Snyder in the relativistic case [10]. In the D -dimensional case deformed algebra proposed by Kempf takes the form:

$$\begin{aligned} [X_i, P_j] &= i\hbar(\delta_{ij}(1 + \beta P^2) + \beta' P_i P_j), [P_i, P_j] = 0, \\ [X_i, X_j] &= i\hbar \frac{(2\beta - \beta') + (2\beta + \beta')\beta P^2}{1 + \beta P^2} (P_i X_j - P_j X_i), \end{aligned} \quad (1)$$

where β and β' are the parameters of deformation. We also suppose that parameters of deformation are nonnegative $\beta, \beta' \geq 0$. Having the uncertainty relation one can obtain that minimal length equals $\hbar\sqrt{\beta + \beta'}$. We note that in the special case $2\beta = \beta'$ the position operators in linear approximation over the deformation parameters commute, i.e. $[X_i, X_j] = 0$.

Deformed commutation relations (1) bring new difficulties in quantum mechanics. Only a few problems are known for which the energy spectra have been found exactly. They are one-dimensional

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harmonic oscillator with minimal uncertainty in position [5] and also with minimal uncertainty in position and momentum [11, 12], D -dimensional isotropic harmonic oscillator [13, 14], three-dimensional relativistic Dirac oscillator [15] and one-dimensional Coulomb problem [16].

The hydrogen atom is the key one in modern physics. Hydrogen atom allows not only highly accurate theoretical predictions but it is also well studied experimentally offering the most precisely measured quantities. The hydrogen atom problem in deformed space with minimal length was considered for the first time by Brau in the special case $2\beta = \beta'$ [17]. The general case of deformation $2\beta \neq \beta'$ was investigated in [18]. The authors used perturbation theory and calculated corrections to the energy levels. But the perturbation theory proposed by the authors did not allow to obtain corrections to the s -levels. To avoid this problem the authors used the numerical methods and cut-off procedure. In our work [19] we developed the modified perturbation theory enabling to calculate the corrections for arbitrary energy levels in hydrogen atom including s -levels. In [20] we applied the modified theory for finding the corrections to the ns -levels in the hydrogen atom. The hydrogen atom in a deformed space was also considered in [9]. In work [21] was considered the elastic scattering problem on the Yukawa and Coulomb potential in deformed space with minimal length.

In this work we proceed the examination of hydrogen atom in deformed space with minimal length. Using the results obtained in works [19, 21] we investigate the orbital magnetic moment of the electron. This paper is organized as follows. In the second section we obtain the continuity equation for the particle in the Coulomb field taking into account some of the results of work [21]. In the third section we consider the corrections to the wave function of the hydrogen atom in a deformed space. In the fourth section we calculate the orbital magnetic moment and compare our results with relativistic corrections. And finally the fifth section contains the discussion.

2 Continuity equation

Here we consider the motion of electron in the Coulomb field of the nucleus. In work [21] we found the continuity equation in a more general case for the particle in Yukawa field. Using the results of work [21] we can immediately write the continuity equation for the Coulomb field. But for the Coulomb potential the calculations are considerably simpler than for Yukawa field and we give these calculations here.

The Hamiltonian for a particle in the external Coulomb field reads

$$H = \frac{\mathbf{P}^2}{2M} - \frac{e^2}{R}, \quad (2)$$

where operators of position X_i and momentum P_i satisfy the deformed commutation relations (1) and

$$R = \sqrt{\sum_{i=1}^3 X_i^2}$$

To construct the continuity equation we write the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \quad (3)$$

One can write the following relation using equation (3)

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar} (\psi^* H\psi - \psi H\psi^*), \quad (4)$$

where $\rho = |\psi|^2$.

To construct the continuity equation it is necessary to use the representation of the operators of positions and momenta that satisfy the deformed commutation relations (1). The momentum representation for such an algebra is well known, but it is not convenient for us. We use the following representation that obeys algebra (1) in the first order over β, β'

$$\begin{cases} X_i = x_i + \frac{2\beta - \beta'}{4} (x_i p^2 + p^2 x_i), \\ P_i = p_i + \frac{\beta'}{2} p_i p^2; \end{cases} \quad (5)$$

where $p^2 = \sum_{j=1}^3 p_j^2$ and operators x_i, p_j satisfy a canonical commutation relation. The position representation $x_i = x_i, p_j = i\hbar \frac{\partial}{\partial x_j}$ can be taken for the ordinary Heisenberg algebra.

As was shown in work [19] Hamiltonian (2) can be expressed in the following form using representation (5) and taking into account only the first order terms in β, β'

$$H = \frac{p^2}{2M} + \frac{\beta' p^4}{2M} - e^2 \left(\frac{1}{\sqrt{r^2 + b^2}} - \frac{2\beta - \beta'}{4} \left(\frac{1}{r} p^2 + p^2 \frac{1}{r} \right) \right), \quad (6)$$

where $b = \hbar \sqrt{2\beta - \beta'}$.

So we can rewrite equation (4) using the Hamiltonian (6)

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & \frac{1}{i\hbar} \left(\frac{1}{2M} (\psi^* p^2 \psi - \psi p^2 \psi^*) + \frac{\beta'}{2M} (\psi^* p^4 \psi - \psi p^4 \psi^*) + \right. \\ & \left. \frac{(2\beta - \beta')e^2}{4} \left(\psi^* \left(\frac{1}{r} p^2 + p^2 \frac{1}{r} \right) \psi - \psi \left(\frac{1}{r} p^2 + p^2 \frac{1}{r} \right) \psi^* \right) \right), \end{aligned} \quad (7)$$

Then using explicit form for the operator $p^2 = -\hbar^2 \nabla^2$ we write the last equation as follows

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & \frac{1}{i\hbar} \nabla \left(-\frac{\hbar^2}{2M} (\psi^* \nabla \psi - \psi \nabla \psi^*) + \frac{\beta' \hbar^4}{2M} (\psi^* \nabla^3 \psi - \psi \nabla^3 \psi^* - \nabla \psi^* \nabla^2 \psi + \right. \\ & \left. \nabla \psi \nabla^2 \psi^*) - \frac{\hbar^2 e^2 (2\beta - \beta')}{4} \left(\psi^* \left(\frac{1}{r} \nabla + \nabla \frac{1}{r} \right) \psi - \psi \left(\frac{1}{r} \nabla + \nabla \frac{1}{r} \right) \psi^* \right) \right) \end{aligned} \quad (8)$$

which can be represented in the continuity equation form:

$$\frac{\partial \rho}{\partial t} + \text{div} \mathbf{j} = 0, \quad (9)$$

where

$$\begin{aligned} \mathbf{j} = & \frac{1}{i\hbar} \left(-\frac{\hbar^2}{2M} (\psi^* \nabla \psi - \psi \nabla \psi^*) + \frac{\beta' \hbar^4}{2M} (\psi^* \nabla^3 \psi - \psi \nabla^3 \psi^* - \nabla \psi^* \nabla^2 \psi + \right. \\ & \left. \nabla \psi \nabla^2 \psi^*) - \frac{\hbar^2 e^2 (2\beta - \beta')}{4} \left(\psi^* \left(\frac{1}{r} \nabla + \nabla \frac{1}{r} \right) \psi - \psi \left(\frac{1}{r} \nabla + \nabla \frac{1}{r} \right) \psi^* \right) \right) \end{aligned} \quad (10)$$

is the probability density flux and here $\nabla^3 = (\nabla, \nabla) \nabla$.

Expression (10) for the probability density flux in a deformed case for the particle moving in the external Coulomb field is somewhat different from the density flux in the ordinary quantum mechanics. In contrast to the ordinary quantum mechanics in a deformed case we have two additional terms into the continuity equation. One of them is caused by the deformed kinetic energy. The second contribution is caused by the Coulomb field. But one should note that in the special case $2\beta = \beta'$ when the position operators are commutative, i.e. $[X_i, X_j] = 0$, the Coulomb potential does not make any contribution to the continuity equation.

3 Corrections to wave function of the hydrogen atom

When we calculate only the first-order corrections to the energy spectrum we use the eigenfunctions of undeformed quantum problem but for high order corrections to the energy spectrum it is necessary to have the corrections to undeformed wave functions too. In contrast to this when we calculate the averages for the other operators but not for the Hamiltonian we must take into account corrections

to the wave functions in the first order of perturbation. So to obtain the correct expression for the current density for the electron in the hydrogen atom we must take into account also corrections to the eigenfunctions caused by deformation.

The wave function of the perturbed quantum system in the first order takes the form

$$\psi_{(q)}^{(1)} = \psi_{(q)}^{(0)} + \sum_{(q') \neq (q)} \frac{V_{(qq')}}{E_{(q)}^{(0)} - E_{(q')}^{(0)}} \psi_{(q')}^{(0)}, \quad (11)$$

where $\psi_{(q)}^{(0)}$ is the wave function of the unperturbed system, $E_{(q)}^{(0)}$ is the energy of the unperturbed problem, $V_{(qq')}$ is the matrix element for the perturbation operator and (q) is the multiindex.

In our case the unperturbed Hamiltonian is the Hamiltonian of the ordinary hydrogen atom and the perturbation operator takes the form similarly to [19]

$$V = \frac{\beta' p^4}{2M} - e^2 \left(\frac{1}{\sqrt{r^2 + b^2}} - \frac{1}{r} - \frac{2\beta - \beta'}{4} \left(\frac{1}{r} p^2 + p^2 \frac{1}{r} \right) \right). \quad (12)$$

So using the eigenfunction of an ordinary hydrogen atom we calculate matrix elements for the perturbation operator

$$V_{(qq')} = \langle n'l'm' | V | nlm \rangle = \delta_{ll'} \delta_{mm'} \left(2M\beta' E_n^{(0)2} \delta_{n'n} - V_{nn'} \right), \quad (13)$$

where $E_n^{(0)} = -\frac{e^2}{an^2}$ is the energy of an ordinary hydrogen atom and

$$V_{nn'} = \langle n'lm | V | nlm \rangle = -\langle n'lm | \frac{e^2}{\sqrt{r^2 + b^2}} | nlm \rangle + \langle n'lm | \frac{e^2}{r} | nlm \rangle + \quad (14)$$

$$\frac{M}{2} (2\beta + 3\beta') (E_n^{(0)} + E_{n'}^{(0)}) \langle n'lm | \frac{e^2}{r} | nlm \rangle + M(2\beta + \beta') \langle n'lm | \frac{e^4}{r^2} | nlm \rangle.$$

Matrix elements $V_{nn'}$ depend on the orbital quantum number l and do not depend on the magnetic quantum number m .

So the wave function of the hydrogen atom in a deformed space with minimal length takes the following form

$$\psi_{nlm}^{(1)} = \psi_{nlm} + \sum_{n' \neq n} \frac{V_{nn'} \psi_{n'lm}}{E_n^{(0)} - E_{n'}^{(0)}} \quad (15)$$

where $V_{nn'}$ is a matrix element given by expression (14) and $\psi_{nlm} = R_{nl}(r) P_l^{|m|}(\cos \vartheta) e^{im\varphi}$ is the eigenfunction of a undeformed hydrogen atom. We do not give here the explicit expression for matrix elements because they are not used in our calculations.

We would also like to stress that for the excited states with the nonzero orbital quantum number $l \neq 0$ we can use a simpler form for the perturbation operator instead of (12)

$$V = \frac{\beta' p^4}{2M} + \frac{(2\beta - \beta')e^2}{4} \left(\frac{1}{r} p^2 + p^2 \frac{1}{r} + \frac{2\hbar^2}{r^3} \right). \quad (16)$$

As we see this perturbation operator is linear over the deformation parameters so the correction to the wave function will be linear in deformation too. We note that for the s -states we are forced to use perturbation operator (12) because the term proportional to $1/r^3$ in (16) gives a divergent contribution in this case. The magnetic moment has a nonzero value only for excited states. Therefore, for its calculation we can use just (16).

4 Magnetic moment

Having a relation for the probability density flux we can find the magnetic moment of the electron in atom. We calculate the electric current density for the electron in the atom multiplying expression (10) by the electron charge and using the wave function of hydrogen atom (15). Taking into consideration only the first order corrections we obtain

$$\begin{aligned} \mathbf{j}_e = & -\frac{e}{i\hbar} \left(-\frac{\hbar^2}{2M} (\psi_{nlm}^* \nabla \psi_{nlm} - \psi_{nlm} \nabla \psi_{nlm}^*) + \frac{\beta' \hbar^4}{2M} (\psi_{nlm}^* \nabla^3 \psi_{nlm} - \psi_{nlm} \nabla^3 \psi_{nlm}^* - \nabla \psi_{nlm}^* \nabla^2 \psi_{nlm} \right. \\ & + \nabla \psi_{nlm} \nabla^2 \psi_{nlm}^*) - \frac{\hbar^2 e^2 (2\beta - \beta')}{4} \left(\psi_{nlm}^* \left(\frac{1}{r} \nabla + \nabla \frac{1}{r} \right) \psi_{nlm} - \psi_{nlm} \left(\frac{1}{r} \nabla + \nabla \frac{1}{r} \right) \psi_{nlm}^* \right) \\ & \left. - \frac{\hbar^2}{2M} \sum_{n' \neq n} \frac{V_{nn'}}{E_n^{(0)} - E_{n'}^{(0)}} (\psi_{nlm}^* \nabla \psi_{n'lm} + \psi_{n'lm}^* \nabla \psi_{nlm} - \psi_{nlm} \nabla \psi_{n'lm}^* - \psi_{n'lm} \nabla \psi_{nlm}^*) \right). \end{aligned} \quad (17)$$

We choose the electron charge in the form $-e$, where $e = 4.8203 \times 10^{-10}$ is the absolute value of the electron charge.

It is convenient to calculate the component of the current density in the spatial spherical coordinates for which the ∇ -operator takes the following form

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial r} + \mathbf{e}_2 \frac{1}{r} \frac{\partial}{\partial \vartheta} + \mathbf{e}_3 \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \varphi}, \quad (18)$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the unit vectors tangent to the coordinate curves at the given point. Using representation (18) for the components of the density current in the spherical coordinates we have

$$j_r = 0, \quad (19)$$

$$j_\vartheta = 0, \quad (20)$$

$$j_\varphi = -\frac{e\hbar m}{Mr \sin \vartheta} \left(|\psi_{nlm}|^2 + 4\beta' M \psi_{nlm}^* \hat{K} \psi_{nlm} + 2e^2 (2\beta - \beta') M \frac{|\psi_{nlm}|^2}{r} + 2 \sum_{n' \neq n} \frac{V_{nn'} \psi_{n'lm}^* \psi_{nlm}}{E_n^{(0)} - E_{n'}^{(0)}} \right), \quad (21)$$

where $\hat{K} = -\frac{\hbar^2 \nabla^2}{2M}$ is the kinetic energy operator in ordinary quantum mechanics and m is the magnetic quantum number.

Relations (19) and (20) can be obtained immediately when we take into account that functions $R_{nl}(r)$ and $P_l^{|m|}(\cos \vartheta)$ are real. This result coincides with the expression for the components of the current density vector in ordinary quantum mechanics.

Having a relation for the density current we can calculate the magnetic moment caused by them

$$d\mu_z = \frac{1}{c} S dI = \frac{1}{c} \pi j_\varphi r^2 \sin^2 \vartheta d\sigma, \quad (22)$$

where $S = \pi r^2 \sin^2 \vartheta$ is the area of the current loop $dI = j_\varphi d\sigma$.

To obtain the total magnetic moment it is necessary to integrate over all of the current tubes. So we have

$$\mu_z = -\frac{e\hbar m}{2Mc} \left(1 + 4\beta' M \langle \hat{K} \rangle + 2(2\beta - \beta') e^2 M \left\langle \frac{1}{r} \right\rangle \right) \quad (23)$$

We note that corrections to the wave function do not make a contribution to the magnetic moment because in the expression for the density current (21) we have the product of two orthogonal functions and after integration these terms disappear.

As was shown in [5] the angular momentum in deformed space with the minimal length takes the form

$$L_{ij} = \frac{1}{1 + \beta P^2} (P_i X_j - P_j X_i), \quad (24)$$

and operators X_i, P_i satisfy algebra (1).

Substituting representation (5) in (25) and taking into account only the first order terms over the deformation parameters we obtain

$$L_{ij} = p_i x_j - p_j x_i, \quad (25)$$

and thus in this case the deformed angular momentum coincides with an ordinary one.

We know that for the hydrogen atom the following relations $\langle \hat{K} \rangle = \langle \hat{p}^2/2M \rangle = e^2/(2an^2)$ and $\langle e^2/r \rangle = e^2/(an^2) = 2\langle \hat{K} \rangle$ take place therefore we can rewrite expression (23) for the magnetic moment in the form

$$\mu_z = -\frac{e}{2Mc}(1 + 8\beta M \langle \hat{K} \rangle) L_z \quad (26)$$

Substituting the explicit form for the averages and taking that $L_z = \hbar m$ we obtain

$$\mu_z = -\mu_B m \left(1 + 4\beta M \frac{e^2}{an^2} \right) \quad (27)$$

where $\mu_B = e\hbar/(2Mc)$ is the Bohr magneton. We note that correction to the magnetic moment depends only on one parameter of deformation in the presence of two-parametric deformed algebra.

Let us compare our results with the corrections that follows from the relativistic theory. The expression for the operator of an orbital magnetic moment in this case reads [22]

$$\hat{\mu}^{(rel)} = -\frac{e}{2Mc} \frac{Mc^2}{E_p} \hat{L} \quad (28)$$

where \hat{L} is the angular momentum operator and $E_p = \sqrt{M^2 c^4 + \hat{p}^2 c^2}$ is the energy of a relativistic particle. In the weak-relativistic approximation we can decompose the energy of the particle in the series over $1/c^2$. Taking into account only the first order terms we obtain

$$\hat{\mu}^{(rel)} = -\frac{e}{2Mc} \left(1 - \frac{1}{Mc^2} \hat{K} \right) \hat{L}, \quad (29)$$

where $\hat{K} = \hat{p}^2/2M$ is the kinetic energy operator of non-relativistic particle.

The average value of the magnetic moment in a weak-relativistic approximation is

$$\mu_z^{(rel)} = \langle nlm | \hat{\mu}_z^{(rel)} | nlm \rangle = -\frac{e}{2Mc} \left(1 - \frac{1}{Mc^2} \langle \hat{K} \rangle \right) L_z \quad (30)$$

Comparing relations (26) and (30) we see that corrections to the magnetic moment take the similar form in both cases but contrary to deformation the relativity theory gives the corrections with the opposite sign. So the deformation of commutation relations leads to the increase of the magnetic moment. At the same time the relativity theory leads to the decrease of the magnetic moment of the electron.

We rewrite relation (27) using two parameters $\Delta x_{min} = \hbar \sqrt{\beta + \beta'}$ and $\eta = \beta/(\beta + \beta')$ instead of β and β' similarly to [18, 19, 20, 21]

$$\mu_z = \mu_z^{(0)} (1 + \varsigma(\Delta x_{min}, \eta, n)), \quad \mu_z^{(0)} = -\mu_B m \quad (31)$$

where

$$\varsigma(\Delta x_{min}, \eta, n) = \frac{\mu_z - \mu_z^{(0)}}{\mu_z^{(0)}} = \eta \frac{4\Delta x_{min}^2}{a^2 n^2} \quad (32)$$

is the specially introduced function which shows the corrections to the orbital magnetic moment caused by deformation.

Using relation (32) we can numerically estimate corrections to the magnetic moment caused by the minimal length effects. Relation (32) shows that function ς depends on Δx_{min} , η and n . As was shown in [19] the parameter η has a bounded domain of variation: $1/3 \leq \eta \leq 1$. For the minimal

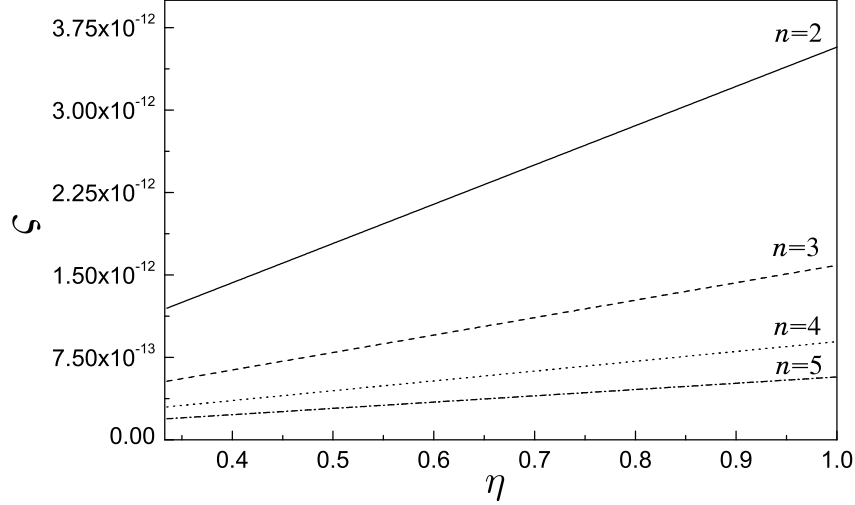


Figure 1: Correction to the orbital magnetic moment of the electron in the hydrogen atom ς as a function of the parameter η for fixed values of the principal quantum number n

length we use the estimation obtained in [18, 19, 20] from the analysis of Lamb shift. Here we take $\Delta x_{min} = 10^{-16}\text{m}$.

Figure 1 shows a dependence of ς as a function of η on its domain of variation when the quantum number n is fixed. We see that correction to the orbital magnetic moment increases with the increasing of η and decreases for higher excited states. We compare the function ς introduced for the estimation of minimal length effects with the relative error of measurements of the Bohr magneton. The most recent measurements show that the relative error for the Bohr magneton equals $\varepsilon = 2.5 \times 10^{-8}$ [23]. Our estimation shows that the upper bound for correction to the orbital magnetic moment equals $\varsigma_{max} = 3.57 \times 10^{-12}$ that is four powers less than the relative error of the corresponding error. So we can conclude that Lamb shift at this moment gives the most precise estimation of minimal length.

5 Discussion

We investigated the orbital magnetic moment of the electron in the hydrogen atom in the deformed space with minimal length. Having an explicit expression for the probability density flux we obtained the electric current density for the electron in the hydrogen atom and calculated corrections to orbital magnetic moment. The orbital magnetic moment of the electron in the hydrogen atom depends only on one deformation parameter in the presence of two-parametric deformation. We showed that orbital magnetic moment in deformed space is proportional to the angular momentum as in an ordinary case. But the factor between the magnetic moment and angular momentum is not constant as in an ordinary quantum mechanics. This factor depends on the deformation parameter β and the mean value of kinetic energy of the electron and increases with the increasing the kinetic energy. The kinetic energy of the electron in the hydrogen atom is inversely proportional to the square of the principal quantum number so the correction to the orbital magnetic moment drops with the increasing of the principal quantum number.

It is interestingly to note that correction to the magnetic moment caused by deformation is similar to the corrections that follow from the relativistic theory. But in contrast to the relativistic theory deformation leads to an opposite sign of the correction. Hence the relativity theory gives a negative correction to the magnetic moment at the same time the deformation of commutation relations leads to a positive one.

In order to estimate the correction to the orbital magnetic moment of the electron caused by deformation we used the upper bound for the minimal length obtained in [18, 19, 20] from the analysis of Lamb shift. We found that the upper bound for a relative correction to the orbital magnetic moment

$\varsigma = (\mu_z - \mu_z^{(0)})/\mu_z^{(0)}$ is $\sim 10^{-12}$. It is four powers lesser than the relative error for the most recent experimental value of Bohr magneton. So we can conclude that at this time the measurements of the Bohr magneton are not enough precise to obtain a more exact upper bound for the minimal length in comparison with the minimal length that follows from the Lamb shift.

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